Some Proofs about Distances and Centers

MCS-236

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Theorem 1 *If G is a graph with radius* $\operatorname{rad} G$ *and diameter* $\operatorname{diam} G$*, then* $\operatorname{rad} G \leq \operatorname{diam} G \leq 2 \operatorname{rad} G$*.*

Proof. Because the radius is the minimum eccentricity of any vertex and the diameter is the maximum, the radius cannot be larger than the diameter.

Let u and v be vertices of G such that $d(u, v) = \operatorname{diam} G$. Let w be a central vertex so that $e(w) = \operatorname{rad} G$. This means that no vertex is at a distance greater than $\operatorname{rad} G$ from w. In particular $d(u, w)$ and $d(v, w)$ are both less than or equal to $\operatorname{rad} G$. Therefore, $d(u, w) + d(v, w) \leq 2 \operatorname{rad} G$. By the triangle inequaltiy, $d(u, v) \leq d(u, w) + d(v, w)$. This establishes that $\operatorname{rad} G \leq \operatorname{diam} G \leq 2 \operatorname{rad} G$. ∎

Theorem 2 *For any graph G, there is some graph H that has G as its center.*

Proof. We can construct H by adding four vertices to G: i_1, i_2, o_1, and o_2. The new edges are $o_1 i_1$, $o_2 i_2$, and for all v in $V(G)$, vi_1 and vi_2. The eccentricity within H of all vertices in $V(G)$ is 2, whereas the eccentricity of the added vertices is 3 for the i vertices and 4 for the o vertices. ∎

Theorem 3 *For a graph G, there exists a graph H that has G as its periphery if and only if all vertices in G have eccentricity 1 or no vertices in G have eccentricity 1.*

Proof. If all vertices in G have eccentricity 1, then G can itself serve as H. On the other hand, if no vertices in G have eccentricity 1, then H can be formed by adding one new vertex, s, and for each vertex v in $V(G)$, the edge sv.

To show the converse, suppose that G has a vertex u that has eccentricity 1, other vertices v and w that have eccentricities greater than 1, and yet G is the periphery of some graph H. We show this leads to a contradiction.

We know that the diameter of G is greater than 1. Because G is an induced subgraph of H, the diameter of H is also greater than 1. Since G is the periphery of H, any vertex in $V(G)$, such as u, must have $e_H(u) = \operatorname{diam} H > 1$. Since $e_G(u) = 1$, there must be some vertex s in $V(H) - V(G)$ that u is farthest from. However, s also has eccentricity equal to $e_H(u)$ yet is not included in the periphery, producing a contradiction.

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